Ch5: Vector space

→ Vector Space

A non emty set of objects (numbers, Functions, Matrices) on wich two operations are defined (addition, Scalar multiplication)

→ Axioms of Addition and scalar multiplication

If the following conditions are satisfies by all object U, V, w in V and all scalars K and J then we call V a vector space and we call the object in V vectors.

↳ Addition:

1. $\vec{U} + \vec{V}$ in V

← اي عند جمع اي عنصرين داخل V يكون الناتج عنصر موجود في V أيضاً

2. $\vec{U} + \vec{V} = \vec{V} + \vec{U}$
3. $\vec{U} + (\vec{V} + \vec{W}) = (\vec{U} + \vec{V}) + \vec{W}$
4. $\vec{U} + \vec{0} = \vec{0} + \vec{U} = \vec{U}$ for all $\vec{U}$ in V
5. $\vec{U} + (-\vec{U}) = \vec{0}$

↳ Scalar multiplication

1. $K\vec{U}$ in V
2. $K(\vec{U} + \vec{V}) = K\vec{U} + K\vec{V}$
3. $(K + J)\vec{V} = K\vec{V} + J\vec{V}$
4. $K(J\vec{V}) = (KJ)\vec{V}$
5. $1\vec{V} = \vec{V}$

Ex1: Is R^n a vector space.

solution:

let $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \quad \vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \quad \vec{x}+\vec{y} = \begin{bmatrix} x_1+y_1 \\ \vdots \\ x_n+y_n \end{bmatrix} = \begin{bmatrix} y_1+x_1 \\ \vdots \\ y_n+x_n \end{bmatrix}$$

$(\vec{x}+\vec{y})+\vec{z} = \vec{x}+(\vec{y}+\vec{z})$

next, we show the existence of an additive identity

Let $\vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ $\quad \vec{x}+\vec{0} = \vec{x}$

Hence $\vec{0}$ is an additive identity

next we prove the existence of an additive inverse

Let $-\vec{x} = \begin{bmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n \end{bmatrix}$ $\quad \vec{x}+(-\vec{x}) = \vec{0}$

Hence $-x$ is an additive inverse

$$a\vec{x} = a\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} ax_1 \\ \vdots \\ ax_n \end{bmatrix}$$

- $a(\vec{x}+\vec{y}) = a\vec{x}+a\vec{y}$
- $(a+b)\vec{x} = a\vec{x}+b\vec{x}$
- $a(b\vec{x}) = (ab)\vec{x}$
- $1\vec{x} = \vec{x}$

Ex2. Vector space of polynomials

Let P_2 be the set of all polynomial of degree 2 as well as the zero polynomial. then P_2 is a vector space.

Solution

We can write $\mathbb{P}_2 = \{a_2x^2 + a_1x + a_0\}$ ($a_i \in \mathbb{R}$ for all i)

To show that $\mathbb{P}_2$ is a vector space, we verify the axioms. Let $p(x)$, $q(x)$ and $r(x)$ be polynomials in $\mathbb{P}_2$ and let a, b, c be polynomials in $\mathbb{P}_2$ and let a, b, c be real numbers. write $p(x) = p_2x^2 + p_1x + p_0$ $q(x) = q_2x^2 + q_1x + q_0$ and $r(x) = r_2x^2 + r_1x + r_0$

- $p(x) + q(x) = (p_2 + q_2)x^2 + (p_1 + q_1)x + (p_0 + q_0) \in \mathbb{P}_2$
- $(p(x) + q(x)) + r(x) = p(x) + (q(x) + r(x))$
- Let $0(x) = 0x^2 + 0x + 0$

  $p(x) + 0(x) = p(x)$ Hence an additive identity exist
- Let $-p(x) = -p_2x^2 - p_1x - p_0$

  $p(x) + (-p(x)) = 0(x)$ Hence an additive inverse exist

→ we need to verify the axioms related to scalar multiplica.

- $a(p(x) + q(x)) = ap(x) + aq(x)$
- $(a+b)p(x) = ap(x) + bq(x)$
- $a(b\,p(x)) = (ab)p(x)$
- $1p(x) = p(x)$

Hence $\mathbb{P}_2$ is vector space.

Ex3: Vector space of Function

Let S be a non empty set and define F_S to be the set of real function on S. $F_S : S \to \mathbb{R}$

Let a, b, c be scalar and f, g, h Functions, the vector operation are defined as.

$$(f+g)(x) = f(x) + g(x)$$
$$(af)(x) = a(f(x))$$

Show that $\mathbb{F}_S$ is a vector space.

Solution:

To verify F_S is a vector space, we must prove the axioms.

Let f, g, h be functions in F_S

- $(f+g)(x) = f(x) + g(x)$
- $(f+g)(x) = (g+f)(x)$
- $((f+g)+h)(x) = (f+g)(x) + h(x) = f(x) + g(x) + h(x)$
$= (f+(g+h))(x)$.

$\Rightarrow (f+g)+h = f+(g+h)$

- Let 0 denote the Function wich is given $0(x) = 0$

Then this is additive identity because

$(f+0)(x) = f(x) + 0(x) = f(x)$ and so $f+0 = f$

- Check for an additive inverse.

Let $-f$ be the Function wich satisfies $(-f)(x) = -f(x)$.

Then $(f+(-f))(x) = f(x) + (-f)(x) = 0$

Hence $f+(-f) = 0$

- Check the axioms for multiplication

$((a+b)f)(x) = (a+b)f(x) = af(x) + bf(x) = (af+bf)(x)$

and so $(a+b)f = af + bf$

- $(a(f+g))(x) = a(f+g)(x) = a(f(x) + g(x)) = af(x) + ag(x)$
$= (af + ag)(x)$ and so $a(f+g) = af + ag$

$((ab)f)(x) = (ab)f(x) = a(bf(x)) = (a(bf))(x)$

so $(abf) = a(bf)$

- $(1f)(x) = 1f(x) = f(x)$ so $1f = f$

→ Theorem:

In any vector space the following are true:

1. $\vec{0}$, the additive identity is unique
2. $-\vec{x}$, the additive inverse is unique
3. $0\vec{x} = \vec{0}$ for all vector $\vec{x}$
4. $(-1)\vec{x} = -x$ for all vector $\vec{x}$

→ Theoreme

Let V be vector space, then $\vec{v} + \vec{w} = \vec{v} + \vec{z}$
$\Rightarrow \vec{w} = \vec{z}$ for all $\vec{v}, \vec{w}, \vec{z} \in V$.

* Spanning sets

→ Subset

Let X and Y be two sets. If all elements of X are also element of Y then we say that X is subset of Y. We write $X \subseteq Y$

→ Linear combination

Let V be a vector space and let $\vec{v_1}, \vec{v_2}, \dots, \vec{v_n} \subseteq V$
A vector $\vec{w} \in V$ is called a linear combination of the v_i if there exists scalars $c_i \in \mathbb{R}$ such that
$\vec{w} = c_1\vec{v_1} + c_2\vec{v_2} + \dots + c_n\vec{v_n}$.

Ex:

Let $\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \subseteq \mathbb{R}^2$

$3\vec{a} - 2\vec{b} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ (linear combination)

→ Span of vector

. Let $S = \{v_1, v_2, ..., v_n\}$ in a vector Space V, then span(S) is the subspace of V that consist of all the possible linear combination of S.

$$Span(S) = c_1\vec{v_1} + c_2\vec{v_2} + ... + c_n\vec{v_n} \quad ; \quad c_i \in \mathbb{R}$$

. Span (S) is always a subspace of V

. If $\vec{w} \in span(S)$, then $\vec{w}$ is a linear combination of S

. If $\vec{w}$ is a linear combination of S, then $\vec{w} \in Span(S)$

↳ Dimension of V :

→ Vectors (R^n)

$dim(R^n) = n \Rightarrow dim(R^2) = 2$

→ Polynomials (P_n)

$dim(P_n) = n+1 \Rightarrow dim(P_2) = 3$

→ matrices (M_{mn})

$dim(M_{m \times n}) = m * n \Rightarrow dim(M_{2 \times 2}) = 4$

Ex1 : Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Determine if A and B are in span $\{M_1, M_2\}$

$= Span \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} ; \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

we want to see if s, t scalars such that

$A = sM_1 + tM_2$

Solution :

For A :

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = s \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + t \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The solution to this equation is given by : $s = 1$, $t = 2$

A is in Spam $\{M_1, M_2\}$

consider B : $B = sM_1 + tM_2$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = s \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + t \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

no value of s and t can found

B is not in span $\{M_1, M_2\}$

Ex2: (Polynomial Span)

Show that $p(x) = 7x^2 + 4x - 3$ is span $\{4x^2 + x, x^2 - 2x + 3\}$

Solution:

$7x^2 + 4x - 3 = a(4x^2 + x) + b(x^2 - 2x + 3)$

1ère methode:

$$\begin{cases} 4a + b = 7 \\ a - 2b = 4 \\ 3b = -3 \end{cases} \Rightarrow \begin{array}{l} a = 2 \\ b = -1 \end{array}$$

$\Rightarrow 7x^2 + 4x - 3 = 2(4x^2 + x) - (x^2 - 2x + 3)$

2ème methode

$$\begin{array}{c} a \\ x \\ x^2 \end{array} \left[\begin{array}{cc|c} 0 & 3 & -3 \\ 1 & -2 & 4 \\ 4 & 1 & 7 \end{array}\right] \quad R_1 \leftrightarrow R_2 \quad \left[\begin{array}{cc|c} 1 & -2 & 4 \\ 0 & 3 & -3 \\ 4 & 1 & 7 \end{array}\right]$$

$$R_2 \leftrightarrow R_3 \quad \left[\begin{array}{cc|c} 1 & -2 & 4 \\ 4 & 1 & 7 \\ 0 & 3 & -3 \end{array}\right] \quad R_2 - 4R_1 \to R_2 \quad \left[\begin{array}{cc|c} 1 & -2 & 4 \\ 0 & 9 & -9 \\ 0 & 3 & -3 \end{array}\right]$$

$$R_2/9 \quad \left[\begin{array}{cc|c} 1 & -2 & 4 \\ 0 & 1 & -1 \\ 0 & 3 & -3 \end{array}\right] \quad R_3 - 3R_2 \to R_3 \quad \left[\begin{array}{cc|c} 1 & -2 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

$$R_1 + 2R_2 \to R_1 \quad \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

$a = 2$

$b = -1$

Ex3: $S\{x^2+1\,;\, x-2\,;\, 2x^2-x\}$ Show that S is a spanning set for $\mathbb{P}_2$, the set of all polynomial degree

Solution:

$p(x) = ax^2 + bx + c$

$p(x) = ax^2 + bx + c = r(x^2+1) + s(x-2) + t(2x^2-x)$

$ax^2 + bx + c = (r+2t)x^2 + (s-t)x + (r-2s)$

$a = r + 2t$

$b = s - t$

$c = r - 2s$

to check that a solution exist, set up the augmented matrix and row reduce

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 0 & 1 & -1 & b \\ 1 & -2 & 0 & c \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2}a + b + \frac{1}{2}c \\ 0 & 1 & 0 & \frac{1}{4}a + \frac{1}{2}b - \frac{1}{4}c \\ 0 & 0 & 1 & \frac{1}{4}a - \frac{1}{2}b - \frac{1}{4}c \end{array}\right]$$

Hence S is a spanning set for $\mathbb{P}_2$

* Linear Independence

→ Linear Independence

Let V be a vector space. If $\{\vec{v_1}, \dots, \vec{v_n}\} \subseteq V$

Then it is linearly independent if :

$$\sum_{i=1}^{n} a_i \vec{v_i} = \vec{0} \quad \text{implies} \quad a_1 = \dots = a_n = 0$$

if otherwise it is linearly dependent.

Ex1: (Linear independence)

Let $S \subseteq \mathbb{P}_2$ be a set of polynomial given by

$S = \{x^2 + 2x - 1, 2x^2 - x + 3\}$

Determine if S is linearly independent

Solution

$a(x^2 + 2x - 1) + b(2x^2 - x + 3) = 0$

$a + 2b = 0$

$2a - b = 0$

$-a + 3b = 0$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & -1 & 0 \\ -1 & 3 & 0 \end{array}\right]$$

$$\xrightarrow{\text{R.E.F}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Hence the solution: $a = b = 0$ and S is linearly independent

Ex2: (Dependent Set)

Determine if the set S given below is independent

$$S = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \right\}$$

Solution :

$$a\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & 1 & 5 & 0 \end{array}\right] \xrightarrow{R.E.F} \quad a = 2, \ b = 3, \ c = -1$$

S is linear dependent

Ex3: $R = \{ 2\vec{u} - \vec{w}, \ \vec{w} + \vec{v}, \ 3\vec{v} + \frac{1}{2}\vec{u} \}$

is linearly independent?

Solution :

$$a(2\vec{u} - \vec{w}) + b(\vec{w} + \vec{v}) + c(3\vec{v} + \tfrac{1}{2}\vec{u}) = 0$$

$$2a + \tfrac{1}{2}c = 0$$
$$-a + b = 0$$
$$b + 3c = 0$$

$$\left[\begin{array}{ccc|c} 2 & 0 & \frac{1}{2} & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right]$$

$$\xrightarrow{R.E.F} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \quad a = b = c = 0$$

R is linearly independent

* Subspaces

Let V be a vector space and $U \subset V$.
We call U a subspace of V if:

. U is nonempty $U \neq \{\varnothing\}$
. Let $\vec{u}, \vec{v} \in U$: $\vec{u} + \vec{v} \in U$
. Let a scalar : $a\vec{u} \in U$.

Ex1:
$V = \mathbb{R}^3 = \{(a,b,c) \,;\, a,b,c \in \mathbb{R}\}$
$U = \{(a,b,0) \,;\, a,b \in \mathbb{R}\}$
Proof that U is subspace of V

. $0 = (0,0,0) \in U$
. Let $\vec{u} = (a,b,0) \in U$
$\vec{v} = (a',b',0) \in U$

$\vec{u} + \vec{v} = (\overset{\in\mathbb{R}}{a+a'}, \overset{\in\mathbb{R}}{b+b'}, 0) \in U$

. $K\vec{U} = (\overset{\in\mathbb{R}}{Ka}, \overset{\in\mathbb{R}}{Kb}, 0) \in U$

$\Rightarrow$ U is a subspace of V

N.B !! . Span of V is always a subspace of V.

Ex2:

Let $M = \{(a,b,c\} : c = 2a - b , a,b \in \mathbb{R}$.
Is M subspace of $\mathbb{R}^3$?

Let $\vec{u} = (a_1 ; b_1 ; 2a_1 - b_1)$
$\vec{v} = (a_2 ; b_2 ; 2a_2 - b_2)$ $\vec{u}, \vec{v} \in M$

. $0 = (0,0,0) \in U$
. $\vec{u} + \vec{v} = (a_1 + a_2 , b_1 + b_2 ; 2(a_1 + a_2) - (b_1 + b_2))$
$\vec{u} + \vec{v} \in M$
. $K\vec{u} = (Ka_1 , Kb_1 , 2Ka_1 - Kb_1) \in M$

so M is a subspace of $\mathbb{R}^3$

→ Basis
We say S is Basis for V if :
1. S is linearly Independent
2. S spans V.

→ Dimension
1. nb of vectors in a basis of V
or
2. nb of free parameters in V (عدد المتغيرات)

Ex₁: Let $w \{(a,b,c,d) : a = 2b , c = 3d\}$
Find dim(w) and basis / systeme of generator of w
$w(2b, b, 3d, d)$ ← يوجد متغيران
$\Rightarrow \dim(w) = 2$
$(2b, b, 3d, d) = b(2,1,0,0) + d(0,0,3,1)$

$\Rightarrow$ basis : $\{(2,1,0,0) , (0,0,3,1)\}$

→ System of generator

To check if a given elements form the systeme of generator of V

$\Rightarrow$ rank (A)

A: contain the element

if rank (A) = nb of element $\Rightarrow$ these element form the systeme of generator

if rank (A) < nb of element $\Rightarrow$ these element doesn't form the systeme of generator of V.

Ex: $x_1 = (1, 2, 1)$ $x_2 = (-2, 3, 0)$ $x_3 = (-1, 5, 1)$

$$\begin{bmatrix} 1 & 2 & 1 \\ -2 & 3 & 0 \\ -1 & 5 & 1 \end{bmatrix} \xrightarrow{R.E.F} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 7 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

rank = 2 < nb of vector (3)

$\Rightarrow$ these element don't form the systeme of generator of $\mathbb{R}^3$

→ Direct sum of vector spaces

Let U and W be subspaces of V. Then V is said to be the direct sum of U and W and we write $V = U \oplus W$

if;
1) $V = U + W$ $\{u + w \,/\, u \in U, w \in W\}$
2) $U \cap W = 0$

__Ex__: Consider the vector space $V = \mathbb{R}^3$

Let $S = \{(x, y, 0) \; x, y \in \mathbb{R}\}$

$T = \{(x, 0, z) \; x, z \in \mathbb{R}\}$

1) prove that S and T are subspaces in $\mathbb{R}^3$
2) prove that $V = S \oplus T$

1) $S = \{ (x, y, 0) \; x, y \in \mathbb{R} \}$

Let $\vec{u} = (x, y, 0)$ and $\vec{v} = (x', y', 0)$
$u, v \in S$;

. $0 = (0, 0, 0) \in S$

. $\vec{u} + \vec{v} = (\underbrace{x + x'}_{\in \mathbb{R}}, \underbrace{y + y'}_{\in \mathbb{R}}, 0) \in S$

. $K\vec{u} = (\underbrace{Kx}_{\in \mathbb{R}}, \underbrace{Ky}_{\in \mathbb{R}}, 0) \in S$

$\Rightarrow$ So S is a subspace of $\mathbb{R}^3$

$T = \{ (x, 0, z) \; x, z \in \mathbb{R} \}$

Let $u = (x, 0, z) \in T$
$v = (x', 0, z') \in T$

. $0 = (0, 0, 0) \in T \Rightarrow T \neq \{\emptyset\}$

. $u + v = (\overset{\in \mathbb{R}}{x + x'}, 0, \overset{\in \mathbb{R}}{z + z'}) \in T$

. $Ku = (Kx, \cancel{0}, Kz) \in T$

$\Rightarrow$ So T is a subspace of $\mathbb{R}^3$

2) Let $x = (x, y, z) \in \mathbb{R}^3$

. $V = (x, y, z) = \overset{\in S}{(x, y, 0)} + \overset{\in T}{(0, 0, z)} \in S + T$

. $S \cap T = \{0\}$?

. $S \cap T = (x, y, 0) \cap (x, 0, z) = (x, 0, 0)$
$\neq (0, 0, 0)$